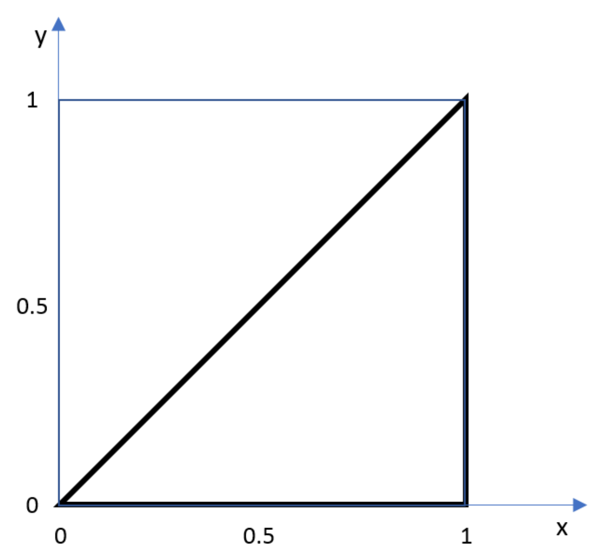
Making a Triangle

Two points are selected on a straight stick by taking independent samples from a uniform probability distribution on the length of the stick. If the stick is broken at those two points, what is the probability that the resulting three pieces can form a triangle?

Solution to Making a Triangle

A necessary and sufficient condition for three line segments to be able to form a triangle is that the length of each segment is less than half of the combined lengths of the three segments.

Let us portray the sample space of the two break points as a square in the x-y plane with one corner at the origin and with sides equal to the length of the original stick. For convenience, we will use a scale such that each side of the square is one unit long. The probability distribution over this sample space is uniform. Let us first consider the half of the sample space such that $x > y$ (shown as a triangle in the figure below).



Given that $x > y$, the lengths of the three segments are:

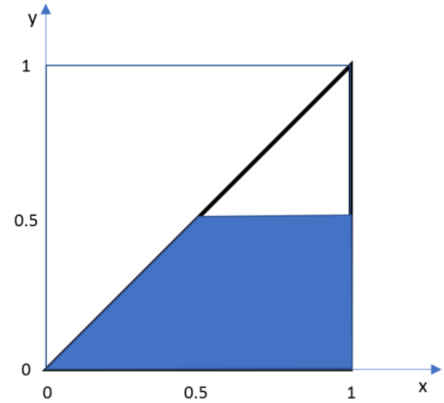
$L_1 = y$

$L_2 = x - y$

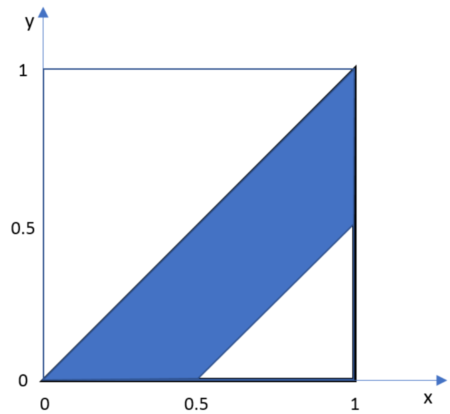
$L_3 = 1 - x$

The line segments can form a triangle if and only if each of these lengths is less than 0.5.

Each of these conditions can be mapped to the sample space, as follows:

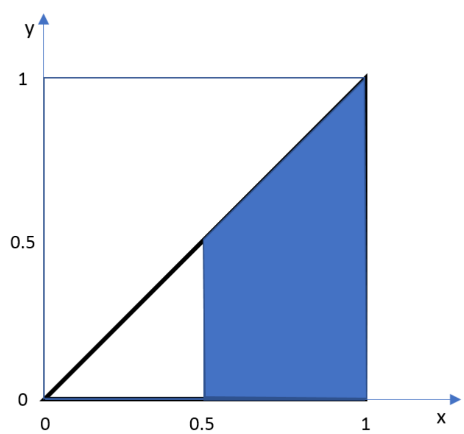


Condition 1: $L_1 < 0.5$ => $y < 0.5$



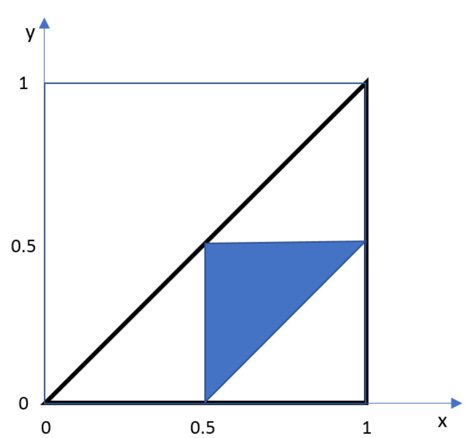
Condition 2: $L_2 < 0.5$ => $x - y < 0.5$

=> $y > x - 0.5$



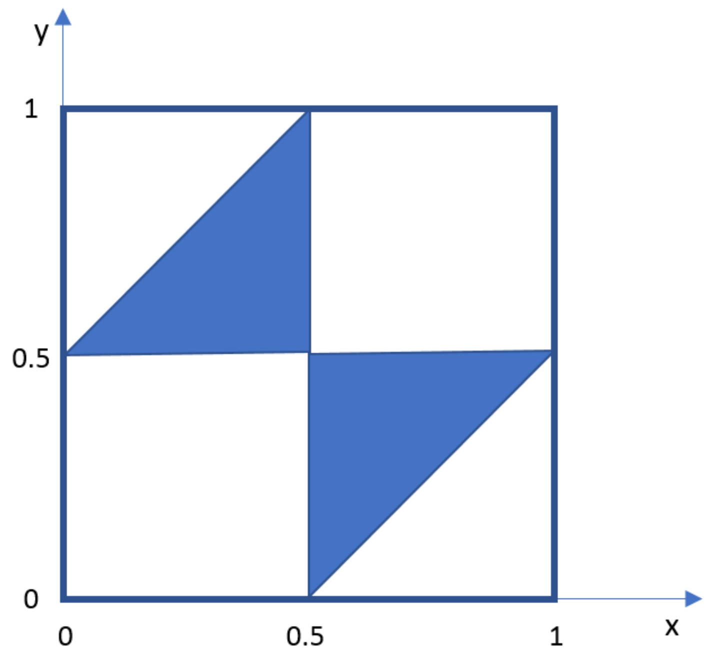
Condition 3: $L_3 < 0.5$ => $1 - x < 0.5$

=> $x > 0.5$



All three conditions combined is shown as the intersection of these three portions of the sample space.

By symmetry, the portion of the sample space that satisfies all three conditions when x < y is a reflection of the portion when x > y. So, the portion of the sample space corresponding to the resulting three line segments being able to make a triangle is shown in the figure below. The probability of a point being in that portion of the sample space is 25%.



Editors' note: This brain teaser was published in Martin Gardner's column in the Scientific American more than 50 years ago.